Inferentialism, Proof-theoretic Semantics, and Computation

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David Pym

UCL Computer Science and Philosophy & Institute of Philosophy
University of London

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https://www.cantab.net/users/david.pym/
david.pym@cantab.net

Overview

- Peter (so many friends and colleagues over the years have been called Peter)
- Some thoughts on Peter's perspectives and priorities
- An approach to the foundations of logic — called *proof-theoretic* semantics — of which I'd hope Peter would approve

Peter

- I had the great privilege as a young scholar of knowing Peter for a few years at Queen Mary
- He was kind, inspiring, and great fun
- Another Peter at Queen Mary at that time was Peter Johnson, Professor of HCI
- Peter and Hilary story ... .
- A warm and generous person. He was always kind to me, and patient with the ignorance and arrogance of a young man from Derbyshire via Cambridge and Edinburgh

Some thoughts on Peter's perspectives and priorities

Reflecting on some of Peter's perspectives and priorities teaches us a few things:

- From ISWIM, we can see a desire to treat languages systematically (and operationally)
- The importance of clean operational semantics (the SECD machine)
- Integration of the operational and denotational (the J-operator)

Aside: Robin Milner used to teach (at Edinburgh) a beautiful course called 'Language Semantics and Implementation', connecting SECD, graph reduction, and SOS — Peter's influence all over it

Inferentialism

- What is inferentialism? Books by Robert Brandom:
  - Making it Explicit
  - Articulating Reasons
  - Reasons for Logic, Logic for Reasons
- How does operational semantics fit in?
- What is the logical set-up more generally?

Proof-theoretic semantics: a new foundation for logic and computation

- Formal reasoning is traditionally justified in terms of model-theoretic truth, in the sense of Tarski:

$$\Gamma \models_{\mathcal{M}} \phi \quad \text{iff} \quad \mathcal{M} \models \Gamma \text{ implies } \mathcal{M} \models \phi$$

- This approach we call model-theoretic semantics (M-tS)
- Here, according to Tarski, semantic consequence amounts to the transmission of truth in model, which is abstract mathematical structure, in general of essentially unconstrained complexity
- Moreover, this is used to justify the correctness of proof systems — this can be seen as highly problematic, especially in informatics
- By contrast, in proof-theoretic semantics, meaning is given in terms of inference
- Let's explore this idea

Proof-theoretic semantics

Proof-theoretic semantics is developed in two main directions:

- Proof-theoretic validity (P-tV), as initiated by Prawitz, Dummett, Sundholm, and others, which is concerned with how we judge proofs (and so proof systems) to be valid
- Base-extension semantics (B-eS), which is concerned with the validity (i.e., semantics) of propositions relative to 'bases' (the counterpart in P-tS to theories in M-tS, kind of) of atomic rules

It can be shown that, in a specific sense, B-eS is the more general perspective. This is slightly surprising, but too much for today (Gheorghiu and Pym 2023)

Proof-theoretic validity (P-tV)

- The validity of a proof in a system, such as Gentzen's natural deduction system NJ, can be judged relative to model-theoretic validity; for example,

$$\Gamma \vdash_{NJ} \phi \quad \text{iff} \quad \text{for all Kripke models } \mathcal{M}, \Gamma \models_{\mathcal{M}} \phi$$

- Proofs can also be interpreted functionally, as in BHK semantics or the 'propositions-as-types' interpretation
- Proof-theoretic validity generalizes this picture by asking how (closed, assumption-free) proofs can be generated from bases of atomic rules (e.g., production rules over atomic formulae)

$$\frac{p_1 \ldots p_k}{p}$$

- There is a very nice introduction to this by Peter Schroeder-Heister in Logica Hejnice, 2007
- This turns out be a very nice place to interpret Milner's notion of *tactical proof* (Phil. Proc. Roy. Soc. A, 1984) — see forthcoming Gheorghiu & Pym article in *Topoi*

Bases (atomic rules)

- Simplest case

$$\frac{p_1 \ldots p_k}{p}$$

For example, with apologies to Tammy,

$$\frac{\text{Luna is a fox} \qquad \text{Luna is female}}{\text{Luna is a vixen}}$$

$$\frac{\text{Luna is a vixen}}{\text{Luna is female}} \qquad \frac{\text{Luna is a vixen}}{\text{Luna is a fox}}$$

- Interesting case:

$$\frac{\begin{matrix} [P_1] & & [P_n] \\ q_1 & \ldots & q_n \end{matrix}}{r}$$

with dischargeable hypotheses — we'll return to this later

- Lots of other cases, with subtle choices

Base-extension semantics (B-eS)

I'll say no more about P-tV for now. Base-extension semantics (B-eS) is a lot more approachable

The basic set-up resembles the satisfaction relations of model-theoretic semantics:

- A base case:

$$\Vdash_{\mathcal{B}} p \quad \text{iff} \quad \vdash_{\mathcal{B}} p \quad (\text{cf.} \quad w \models p \quad \text{iff} \quad w \in V(p))$$

- Inductive cases for the connectives: for example,

$$\Vdash_{\mathcal{B}} \phi \wedge \psi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \phi \text{ and } \Vdash_{\mathcal{B}} \psi$$

But this set-up has some deceptively deep consequences. Let's look at Be-S for intuitionistic propositional logic

It reveals some deep issues about the meaning of disjunction

Sandqvist's Semantics for IPL

Base rules $\mathcal{R}$, application of base rules, and satisfaction of formulae in a (possibly finite) countable base $\mathcal{B}$ of rules $\mathcal{R}$ are defined as follows:

$$\dfrac{\begin{matrix}[P_1] & & [P_n] \\ q_1 & \ldots & q_n\end{matrix}}{r}\,\mathcal{R}$$

(Ref) $P, p \vdash_{\mathcal{B}} p$

(App$_{\mathcal{R}}$) if $((P_1 \Rightarrow q_1), \ldots, (P_n \Rightarrow q_n)) \Rightarrow r)$ and, for all $i \in [1, n]$, $P, P_i \vdash_{\mathcal{B}} q_i$, then $P \vdash_{\mathcal{B}} r$

(At) for atomic p, $\Vdash_{\mathcal{B}} p$ iff $\vdash_{\mathcal{B}} p$

($\supset$) $\Vdash_{\mathcal{B}} \phi \supset \psi$ iff $\phi \Vdash_{\mathcal{B}} \psi$

($\wedge$) $\Vdash_{\mathcal{B}} \phi \wedge \psi$ iff $\Vdash_{\mathcal{B}} \phi$ and $\Vdash_{\mathcal{B}} \psi$

($\vee$) $\Vdash_{\mathcal{B}} \phi \vee \psi$ iff, for every atomic p and every $\mathcal{C} \supseteq \mathcal{B}$, if $\phi \Vdash_{\mathcal{C}} p$ and $\psi \Vdash_{\mathcal{C}} p$, then $\Vdash_{\mathcal{C}} p$

($\bot$) $\Vdash_{\mathcal{B}} \bot$ iff, for all atomic p, $\Vdash_{\mathcal{B}} p$

(Inf) for $\Theta \neq \emptyset$, $\Theta \Vdash_{\mathcal{B}} \phi$ iff, for every $\mathcal{C} \supseteq \mathcal{B}$, if $\Vdash_{\mathcal{C}} \theta$ for every $\theta \in \Theta$, then $\Vdash_{\mathcal{C}} \phi$

There is a substitution (cut) operation on bases that maps derivations $P \vdash_{\mathcal{B}} p$ and $p, Q \vdash_{\mathcal{B}} q$ to a derivation $P, Q \vdash_{\mathcal{B}} q$.

- Gentzen's natural deduction calculus NJ — as studied extension by Prawitz, Martin-Löf, and so on — is sound and complete for Sandqvist's semantics
- But the completeness result depends critically on the correspondence between the clause for $\vee$ and the $\vee$-elimination rule of NJ ...

Disjunction

- The *semantic* clause for disjunction is very proof-theoretic. In fact, it corresponds to (i) the second-order propositional definition of $\vee$ or, alternatively, (ii) to the natural deduction elimination rule:

$$\dfrac{\phi \vee \psi \qquad \begin{matrix}[\phi] \\ \vdots \\ p\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ p\end{matrix}}{p}$$

- Why? Because taking $\Vdash_{\mathcal{B}}$ iff $\Vdash_{\mathcal{B}} \phi$ or $\Vdash_{\mathcal{B}} \psi$ leads to *incompleteness* (Piecha and Schroeder-Heister)
- More on issues around disjunction in, for example,
  - Neil Tennant's work
  - Pym, Ritter, and Robinson, *Studia Logica*, 2024, for a category-theoretic perspective

Why is this?

- From the inferentialist perspective, Kripke's clause is too strong because it assumes that the suasive content of a disjunction is identical to that of its disjuncts
- However, Sandqvist's treatment, corresponding to the $\vee$-elimination rule of NJ for disjunction, expresses that whatever can be inferred from both disjuncts can be inferred from the disjunction. It is this correspondence that allows completeness to go through
- Unless, following recent work of Nascimento and Stafford or earlier work of Goldfarb, one encodes Kripke models in B-eS

Our Developments of B-eS

- Exploring a much wider range of logical systems than previously considered
- Foundational issues

Substructural and Modal Logics

We — that is, my group around UCL — have developed B-eS for

- Substructural logics lack the familiar rules of weakening (adding assumptions is allowed) and contraction (removing duplicate assumptions is allowed)
- Substructural logics: IMLL and BI — BI is the foundation of ‘Separation Logic’, the formal theory of memory cells and pointers represented as a theory of boolean BI
- The ‘exponentials’ of Linear Logic have been a challenge — excellent work by Yll Buzoku and Victor Nascimento
- Modal logics: K, T, K4, S4, with S5 and DEL-like logics in development

All these logics require working with richer relational structures on bases — echoes of Kripke semantics for non-classical logics

Category-theoretic Foundations

More foundationally, we have

- A category-theoretic treatment of Sandqvist's set-up, establishing the formal naturality of all the constructions, important for theoretical development.
- Makes clear how the issues around disjunction work
- Also get a topological account of the issues around the semantics of disjunction
- Extensions to substructural logics?
  - Start with IMLL ($\otimes$ and $\multimap$) — done
  - Combine IMLL and IPL to get BI — at the level of proof *systems*, this is well understood category-theoretically — in progress, very general approach

Joint work involving also Tao Gu, Eike Ritter, and Edmund Robinson (who was last year's Landin lecturer, and is also from Derbyshire)

Connections to Logic Programming (starting from old work of Miller)

- We have established a deep connection between Sandqvist's completeness theorem and the least fixed semantics of logic programming — the T^{ω} least fixed point semantics of analytic resolution in hereditary Harrop formulae
- hHfs:

$$\begin{array}{llll} \text{Definite formulae} & D & := & A \mid G \supset A \mid D \wedge D \\ \text{Goal formulae} & G & := & A \mid D \supset G \mid G \wedge G \mid G \vee G \end{array}$$

  Write $\mathcal{P}$ for sets of definite formulae (aka logic programs)
- All the classes of base rules consider in P-tS live inside a fragment ('atomic inferences') of Definite formulae
- It turns out that completeness in Be-S can be understood from this perspective

How it works: operationally

- We read and apply rules from conclusion to premisses
- We consider *uniform proofs* of hHfs (for which they are complete)
- Right rules are always preferred to left rules
- So, faced with $\mathcal{P} \vdash G$, we first reduce the G until we have only atoms on the right
- Definite formulae of the form $G \supset A$ drive *resolution* via the $\supset L$ rule of the sequent calculus LJ

How it works: operationally

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\mathcal{P} \vdash G'} \qquad \dfrac{}{A' \vdash A'}\,Ax}{\mathcal{P} \vdash A'}\,G' \supset A' \in \mathcal{P}}{\vdots\ \begin{array}{l}\text{right rules}\\ \text{reduce } G\end{array}}}{\mathcal{P} \vdash G} \qquad \dfrac{}{A \vdash A}\,Ax}{\mathcal{P} \vdash A}\; G \supset A \in \mathcal{P}$$

Resolution, that is uniform proofs with just the resolution rule (essentially, $\supset L$) the only left rule, can be seen as being complete for hHfs (assuming $\wedge$s always removed, written $[\mathcal{P}]$)

How it works: denotationally

- Define an Herbrand interpretation as a function $I : \mathcal{W} \to \wp(\mathcal{H})$ from the set of all programs to the powerset of the Herbrand universe (the set of all atoms)
- Herbrand interpretations form a complete lattice with least element $I_\perp(\mathcal{P}) = \emptyset$
- Define a satisfaction relation $I, \mathcal{P} \models G$, which is Kripke except for $I, \mathcal{P} \models D \supset G$ iff $I, \mathcal{P} \cup [\mathcal{D}] \models G$
- Define an operator T on Herbrand interpretations as follows

  $$T(I)(\mathcal{P}) := \{ \ A \mid A \in [\mathcal{P}] \text{ or there is } G \supset A \in [\mathcal{P}] \text{ s.t. } I, \mathcal{P} \models G \ \}$$

- This captures resolution semantically (and, in fact, proof-theoretically semantically) through its least fixed point (guaranteed to exist by Tarski's theorem)

How it works: back to P-tS

The detailed argument is in A. Gheorghiu and D. Pym, Definite formulae, Negation-as-Failure, and the Base-extension Semantics of Intuitionistic Propositional Logic, *Bulletin of the Section of Logic*, Czech Academy of Sciences, 2023.

But here are the key points:

- Sandqvist's completeness theorem works by reducing formulae to a 'special base' atoms in a structure-preserving way that respects satisfaction
- The base rules are defined in a *fragment of definite formulae*
- So we can use the resolution engine, which also respects satisfaction, to construct the special base
- This also establishes a connection with Kripke semantics while retaining the inferential basis, and gives more structure to the special base, also within definite formulae

How it works: back to P-tS

- A base rule

$$\frac{p_1 \ldots p_n}{p}$$

corresponds to a formula

$$(p_1 \wedge \ldots \wedge p_n) \supset p$$

- It's easy to see that this formula lies with the class of definite formulae (aka logic programs) since goal formulae include conjunctions of atoms

Reductive Logic and P-tS

This specific set-up, although I think pretty enough in itself, is really just a foothill in a much bigger story:

- Consider *reductive logic* as opposed to *deductive logic*
- That is, as we have seen with uniform proof for hHfs, we read rules from conclusion to premisses: not

$$\frac{\text{Established Premiss}_1 \ldots \text{Established Premiss}_n}{\text{Conclusion}} \Downarrow$$

  but instead

$$\frac{\text{Sufficient Premiss}_1 \ldots \text{Sufficient Premiss}_n}{\text{Putative Conclusion}} \Uparrow$$

- Compare with model-theoretic clauses, which can also be read reductively

Some key points

- The space of constructions encountered in reductive logic is bigger than, but contains, the space of proofs in the underlying logic — this raises challenges
- This space corresponds more-or-less exactly to the space of proof-like constructions that underpins P-tV — but that's a whole story in itself, which we can also handle category-theoretically (using polynomial functors, as in my book with Ritter)
- The space of constructions is explored by proof-search — again, as in uniform proofs for hHfs
- Proof-search is inherently stateful computation
- Co-algebra is a good tool for this — see Gheorghiu, Docherty, and Pym in recent Abramsky Festschrift

Reductive Logic and P-tS

- *Pace* my own book (with Ritter) on reductive logic, there is not really a 'mathematical reductive logic' or a philosophy of reductive logic
- We are building them
- First steps already taken (published or submitted):
  - P-tS (in fact P-tV) semantics for Milner's notion of tactical proof, which is the basis for most theorem provers developed since the late 20th Century, and which let to the programming language ML and its derivatives
  - Connections between B-eS and co-algebraic approaches to reductive logic, starting from logic programming
  - A new approach to completeness theorems via the semantics of proof-search
- Enough

Towards an inferentialist philosophy of information

- Logic's account of the semantics of information is woefully inadequate: even Johan van Benthem agrees with me on this
- Lots of 'information interpretations' of non-classical logics, but it's all a bit after-the-fact, to put it politely, even the stuff that comes from informatics
- This is a serious issue in informatics: program and system correctness, meaning/properties of computational/mathematical models: do we really want completeness (say) to rely on syntactic instances of more-or-less arbitrarily large/complex structures in which informal classical mathematical reasoning is used?

Inferentialism and P-tS to the Rescue?

- Can we characterize a semantic notion of information — contrasted with quantitative notions such as Shannon's and the developments of complexity theory — in terms of inference?
- In my judgement, none of the approaches out there is adequate: situation theory's account, Floridi's account, and others
- We are starting to address this:
  - Some technical basics: B-eS for modal logics, K ... $S4$, and $S5$
  - Now Public Announcement Logic ... where Brandom's 'making it explicit' reverberates through the examples
  - Then ... an inferentialist situation theory, 'infons' and all?

Conjecture

An inferentialist reformulation of situation theory that is grounded in base-extension semantics can give an account of information that incorporates all of the key extant, and only weakly connected, approaches:

- *Information-as-range. In this view, information is modelled by the range of possibilities that are consistent with it — Kripke semantics, epistemic logics, etc.*
- *Information-as-correlation. This approach puts what information is about in the focus of the investigation — situation theory, infons, infon logic, etc.*
- *Information-as-code. Here information is thought of as encoded in sentences and messages (cf. Shannon). This allows for a syntactical investigation of information which relies on proof-theory rather than semantics. The inferences possible given a certain message tell us its informational content.*

Executable Models: Towards an Inferentialist View

- System modelling:
  - Executable models — such as discrete-event models, in particular — have an inferentialist semantic interpretation (Kuorikoski and Reijula): each run of the model is an inference if its properties
  - So a model's meaning is the sum of the properties determined by all its executions
  - So we aim — though this is a big project in itself — to set-up the basic execution steps of a model as bases for the meaning of system, supporting a B-eS of system properties:

$$S \stackrel{f(\mathcal{B})}{\rightarrow} T \Vdash_{\mathcal{B}} \phi$$

- Inferentialist resource semantics
- Towards a philosophy of information security: characterizing the basic concepts (cf. Dummett's 'logical basis of metaphysics', a key text in P-tS)

Would Peter approve?

- I certainly hope so
- Why?
- It's actually very simple: meaning is given through the mechanisms by which properties are inferred
- The approach to the semantics of computation that is known as operational semantics, Peter's primary view of the world, lies squarely within the inferentialist position.

Thank you for your kind attention